Please do each of the following problems, providing concise answers with justification.

The total number of points is 100.

1. (10p) Let $A \in \mathcal{C}^{m \times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full column rank.

2. (10p) Let $A \in \mathcal{C}^{m\times m}$ and $A = A^*$.

   (a) Prove that all eigenvalues of A are real.

   (b) Let x and y be eigenvectors of A corresponding to eigenvalues λ and μ respectively. Give a sufficient condition on λ and μ for x and y to be orthogonal.

3. (10p) Prove that the determinant of a Householder reflector is negative one.

4. (15p) Fix $0 < \varepsilon < 1$ and suppose that $A \in R^{m \times m}$ is symmetric and nonsingular. Show that if $\|A - I\|_F \geq \varepsilon$, then $\|A^{-1} - I\|_F \geq \frac{\varepsilon}{2}$, where $\| \cdot \|_F$ denotes the Frobenius norm.

5. (15p) Determine the rate of convergence of the Rayleigh quotient $r(\mathbf{v}_k) = \mathbf{v}_k^T A \mathbf{v}_k$, to an eigenvalue of $A \in \mathcal{R}^{n\times n}, A = A^T$, with vectors $\mathbf{v}_k \in \mathcal{R}^n$ given by the normalized power method $\mathbf{v}_{k+1} = A\mathbf{v}_k / ||A\mathbf{v}_k||$.

6. Suppose $Ax = b$ and $(A + \delta A)\hat{x} = b + \delta b$ and assume that

$$\|(I + A^{-1}\delta A)^{-1}\| \leq \frac{1}{1 - \|\delta A\|\|A^{-1}\|}.$$

(a) (10p) Show that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}} \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right), \tag{1}$$

where $\hat{x} = x + \delta x$ and $\kappa(A) = \|A\| \cdot \|A^{-1}\|$.

(b) (10p) Let $\delta A = 0$ and

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Find vectors δb and b such that the bound (1) becomes an equality.

7. (20p) Let $A \in \mathcal{R}^{n\times n}$ be symmetric and positive definite (spd), and consider the following iteration.

> Choose $A_0 = A$
> **for** $k = 0, 1, 2, \ldots$
>   Compute the Cholesky factor L_k of A_k (so $A_k = L_k L_k^T$)
>   Set $A_{k+1} = L_k^T L_k$
> **end**

Here L_k is lower triangular with positive diagonal elements.

(a) Show that A_k is similar to A, and that A_k is spd (the iteration is therefore well-defined).

Now consider the special case of a 2×2 spd matrix,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad a \geq c,$$

(b) For this matrix, perform one step of the algorithm and write down A_1.

(c) Use the result from **(b)** to argue that A_k converges to $\text{diag}(\lambda_1, \lambda_2)$, where the eigenvalues of A are ordered as $\lambda_1 \geq \lambda_2 > 0$.